A Primer on Index Notation

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1. **Index versus Vector Notation**

   Index notation (a.k.a. Cartesian notation) is a powerful tool for manipulating multidimensional equations. However, there are times when the more conventional vector notation is more useful. It is therefore important to be able to easily convert back and forth between the two. This primer will use both index and vector formulations, and will adhere to the notation conventions summarized below:

	Vector Notation	Index Notation
scalar	a	a
vector	$\vec{a}$	a_i
tensor	$\underline{\underline{A}}$	A_{ij}

   In either notation, we tend to group quantities into one of three categories:

   scalar A magnitude that does not change with a rotation of axes.

   vector Associates a scalar with a direction.

   tensor Associates a vector (or tensor) with a direction.

2. **Free Indices**

   (a) A free index appears once and only once within each additive term in an expression. In the equation below, i is a free index:

   $$a_i = \epsilon_{ijk} b_j c_k + D_{ij} e_j$$

   (b) A free index implies three distinct equations. That is, the free index sequentially assumes the values 1, 2, and 3. Thus,

   $$a_j = b_j + c_j \quad \text{implies} \quad \begin{cases} a_1 = b_1 + c_1 \\ a_2 = b_2 + c_2 \\ a_3 = b_3 + c_3 \end{cases}$$

(c) The same letter must be used for the free index in every additive term. The free index may be renamed if and only if it is renamed in every term.

(d) Terms in an expression may have more than one free index so long as the indices are distinct. For example the vector-notation expression $\underline{\underline{A}} = \underline{\underline{B}}^T$ is written $A_{ij} = (B_{ij})^T = B_{ji}$ in index notation. This expression implies nine distinct equations, since i and j are both free indices.

(e) The number of free indices in a term equals the *rank* of the term:

	Notation	Rank
scalar	a	0
vector	a_i	1
tensor	A_{ij}	2
tensor	A_{ijk}	3

Technically, a scalar is a tensor with rank 0, and a vector is a tensor of rank 1. Tensors may assume a rank of any integer greater than or equal to zero. You may only sum together terms with equal rank.

(f) The first free index in a term corresponds to the row, and the second corresponds to the column. Thus, a vector (which has only one free index) is written as a column of three rows,

$$\vec{a} = a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

and a rank-2 tensor is written as

$$\underline{\underline{A}} = A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

3. **Dummy Indices**

(a) A dummy index appears twice within an additive term of an expression. In the equation below, j and k are both dummy indices:

$$a_i = \epsilon_{ijk} b_j c_k + D_{ij} e_j$$

(b) A dummy index implies a summation over the range of the index:

$$a_{ii} \equiv a_{11} + a_{22} + a_{33}$$

(c) A dummy index may be renamed to any letter not currently being used as a free index (or already in use as another dummy index pair in that term). The dummy index is "local" to an individual additive term. It may be renamed in one term (so long as the renaming doesn't conflict with other indices), and it does not need to be renamed in other terms (and, in fact, may not necessarily even be present in other terms).

4. **The Kronecker Delta**

The Kronecker delta is a rank-2 symmetric tensor defined as follows:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

or,

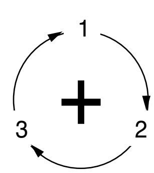
$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. **The Alternating Unit Tensor**

(a) The alternating unit tensor is a rank-3 antisymmetric tensor defined as
follows:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123,\ 231, \text{ or } 312 \\ 0 & \text{if any two indices are the same} \\ -1 & \text{if } ijk = 132,\ 213, \text{ or } 321 \end{cases}$$

The alternating unit tensor is positive when the indices assume any clockwise cyclical progression, as shown in the figure:



(b) The following identity is extremely useful:

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

6. **Commutation and Association in Vector and Index Notation**

   (a) In general, operations in vector notation do *not* have commutative or associative properties. For example,

   $$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$$

   (b) All of the terms in index notation are scalars (although the term may represent multiple scalars in multiple equations), and only multiplication/division and addition/subtraction operations are defined. Therefore, commutative and associative properties hold. Thus,

   $$a_i b_j = b_j a_i$$

   and,

   $$(a_i b_j) c_k = a_i (b_j c_k)$$

   A caveat to the commutative property is that calculus *operators* (discussed later) are not, in general, commutative.

7. **Vector Operations using Index Notation**

   (a) Multiplication of a vector by a scalar:

Vector Notation	Index Notation
$a\vec{b} = \vec{c}$	$ab_i = c_i$

   The index i is a free index in this case.

   (b) Scalar product of two vectors (a.k.a. dot or inner product):

Vector Notation	Index Notation
$\vec{a} \cdot \vec{b} = c$	$a_i b_i = c$

   The index i is a dummy index in this case. The term "scalar product" refers to the fact that the result is a scalar.

   (c) Scalar product of two tensors (a.k.a. inner or dot product):

Vector Notation	Index Notation
$\underline{\underline{A}} : \underline{\underline{B}} = c$	$A_{ij} B_{ji} = c$

   The two dots in the vector notation indicate that both indices are to be summed. Again, the result is a scalar.

(d) Tensor product of two vectors (a.k.a. dyadic product):

Vector Notation	Index Notation
$\vec{a}\vec{b} = \underline{\underline{C}}$	$a_i b_j = C_{ij}$

The term "tensor product" refers to the fact that the result is a tensor.

(e) Tensor product of two tensors:

Vector Notation	Index Notation
$\underline{\underline{A}} \cdot \underline{\underline{B}} = \underline{\underline{C}}$	$A_{ij} B_{jk} = C_{ik}$

The single dot refers to the fact that only the inner index is to be summed. Note that this is *not* an inner product.

(f) Vector product of a tensor and a vector:

Vector Notation	Index Notation
$\vec{a} \cdot \underline{\underline{B}} = \vec{c}$	$a_i B_{ij} = c_j$

Given a unit vector $\hat{n}$, we can form the vector product $\hat{n} \cdot \underline{\underline{B}} = \vec{c}$. In the language of the definition of a tensor, we say here that then tensor $\underline{\underline{B}}$ associates the vector $\vec{c}$ with the direction given by the vector $\hat{n}$. Also, note that $\vec{a} \cdot \underline{\underline{B}} \neq \underline{\underline{B}} \cdot \vec{a}$.

(g) Cross product of two vectors:

Vector Notation	Index Notation
$\vec{a} \times \vec{b} = \vec{c}$	$\epsilon_{ijk} a_j b_k = c_i$

Recall that

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$$

Now, note that the notation $\epsilon_{ijk} a_j b_k$ represents three terms, the first of which is

$$\begin{aligned}
\epsilon_{1jk} a_j b_k &= \epsilon_{11k} a_1 b_k + \epsilon_{12k} a_2 b_k + \epsilon_{13k} a_3 b_k \\
\\
&= \epsilon_{111} a_1 b_1 + \epsilon_{112} a_1 b_2 + \epsilon_{113} a_1 b_3 + \\
&\quad \epsilon_{121} a_2 b_1 + \epsilon_{122} a_2 b_2 + \epsilon_{123} a_2 b_3 + \\
&\quad \epsilon_{131} a_3 b_1 + \epsilon_{132} a_3 b_2 + \epsilon_{133} a_3 b_3 \\
\\
&= \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 \\
&= a_2 b_3 - a_3 b_2
\end{aligned}$$

(h) Contraction or Trace of a tensor (sum of diagonal terms):

Vector Notation	Index Notation
$\mathrm{tr}\left(\underline{\underline{A}}\right) = b$	$A_{ii} = b$

8. **Calculus Operations using Index Notation**

Note: The spatial coordinates (x, y, z) are renamed as follows:

$$\begin{aligned}
x &\rightarrow x_1 \\
y &\rightarrow x_2 \\
z &\rightarrow x_3
\end{aligned}$$

(a) Temporal derivative of a scalar field $\phi(x_1, x_2, x_3, t)$:

$$\frac{\partial \phi}{\partial t} \equiv \partial_0 \phi$$

There is no physical significance to the "0" subscript. Other notation may be used.

(b) Gradient (spatial derivatives) of a scalar field $\phi(x_1, x_2, x_3, t)$:

$$\begin{aligned}
\frac{\partial \phi}{\partial x_1} &\equiv \partial_1 \phi \\
\frac{\partial \phi}{\partial x_2} &\equiv \partial_2 \phi \\
\frac{\partial \phi}{\partial x_3} &\equiv \partial_3 \phi
\end{aligned}$$

These three equations can be written collectively as

$$\frac{\partial \phi}{\partial x_i} \equiv \partial_i \phi$$

In vector notation, $\partial_i \phi$ is written $\nabla \phi$ or grad ϕ. Note that $\partial_i \phi$ is a vector (rank=1). Some equivalent notations for $\partial_i \phi$ are

$$\partial_i \phi \equiv \partial x_i \phi$$

and, occasionally,

$$\partial_i \phi \equiv \partial \phi_{,i}$$

(c) Gradient (spatial derivatives) of a vector field $\vec{a}(x_1, x_2, x_3, t)$:

$$\begin{aligned} \frac{\partial \vec{a}}{\partial x_1} &\equiv \partial_1 a_i \\ \frac{\partial \vec{a}}{\partial x_2} &\equiv \partial_2 a_i \\ \frac{\partial \vec{a}}{\partial x_3} &\equiv \partial_3 a_i \end{aligned}$$

These three equations can be written collectively as

$$\frac{\partial a_i}{\partial x_j} \equiv \partial_j a_i$$

In vector notation, $\partial_j a_i$ is written $\nabla \vec{a}$ or grad $\vec{a}$ Note that $\partial_j a_i$ is a tensor (rank=2):

$$\text{grad } \vec{a} = \frac{\partial a_i}{\partial x_j} = \partial_j a_i = \begin{bmatrix} \partial_1 a_1 & \partial_1 a_2 & \partial_1 a_3 \\ \partial_2 a_1 & \partial_2 a_2 & \partial_2 a_3 \\ \partial_3 a_1 & \partial_3 a_2 & \partial_3 a_3 \end{bmatrix}$$

The index on the denominator of the derivative is the row index. Note that the gradient increases by one the rank of the expression on which it operates.

(d) Divergence of a vector field $\vec{a}(x_1, x_2, x_3, t)$:

$$\text{div } \vec{a} = \nabla \cdot \vec{a} = \partial_i a_i = b$$

Notice that $\partial_i a_i$ is a scalar (rank=0).
Important note: The divergence decreases by one the rank of the expression on which it operates by one. It is not possible to take the divergence of a scalar.

(e) Curl of a vector field $\vec{a}(x_1, x_2, x_3, t)$:

$$\text{curl } \vec{a} = \nabla \times \vec{a} = \epsilon_{ijk} \partial_j a_k = b_i$$

Notice that $\epsilon_{ijk} \partial_j a_k$ is a vector (rank=1).
Important note: The curl does not change the rank of the expression on which it operates. It is not possible to take the curl of a scalar.

(f) Laplacian of a vector field $\vec{a}(x_1, x_2, x_3, t)$:

$$\nabla^2 \vec{a} \equiv \nabla \cdot (\nabla \vec{a}) = \text{div (grad } \vec{a}) = \partial_i \partial_i a_j = b_j$$

9. **The ordering of terms in expression involving calculus operators**

Index notation is used to represent vector (and tensor) quantities in terms of their constitutive scalar components. For example, a_i is the i^{th} component of the vector $\vec{a}$. Thus, a_i is actually a collection of three scalar quantities that collectively represent a vector.

Since index notation represents quantities of all ranks in terms of their scalar components, the order in which these terms are written within an expression is usually unimportant. This differs from vector notation, where the order of terms in an expression is often very important. An extremely important caveat to the above discussion on independence of order is to pay special attention to operators (e.g. div, grad, curl). In particular, remember that the rules of calculus (e.g. product rule, chain rule) still apply.

Example 1:

$$\frac{\partial}{\partial x_k}(a_i b_j) \equiv \partial_k (a_i b_j) = a_i \partial_k b_j + b_j \partial_k a_i \quad \text{(product rule)}$$

Example 2: Show that $\nabla \cdot \vec{a} \neq \vec{a} \cdot \nabla$

$$\nabla \cdot \vec{a} = \partial_i a_i = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = \quad \text{a scalar}$$

whereas

$$\vec{a} \cdot \nabla = a_i \partial_i = a_1 \frac{\partial}{\partial x_1}() + a_2 \frac{\partial}{\partial x_2}() + a_3 \frac{\partial}{\partial x_3}() = \quad \text{an operator}$$

Thus,

$$\partial_i a_i \neq a_i \partial_i$$

10. **Decomposition of a Tensor into Symmetric and Antisymmetric Parts**

(a) A tensor $\underline{\underline{Q}}$ *symmetric* if it is equal to its transpose:

$$Q_{ij} = Q_{ji}$$

(b) A tensor $\underline{\underline{R}}$ *antisymmetric* if it is equal to the negative of its transpose:

$$R_{ij} = -R_{ji}$$

Note that the diagonal terms of R_{ij} must necessarily be zero.

(c) Any arbitrary tensor $\underline{\underline{T}}$ may be decomposed into the sum of a symmetric tensor (denoted $T_{(ij)}$) and an antisymmetric tensor (denoted $T_{[ij]}$).

$$T_{ij} = T_{(ij)} + T_{[ij]}$$

To show this, we start with $\underline{\underline{T}}$ and then add and subtract one-half of its transpose:

$$T_{ij} = T_{ij} + \left(\frac{1}{2}T_{ji} - \frac{1}{2}T_{ji}\right)$$

$$T_{ij} = \left(\frac{1}{2}T_{ij} + \frac{1}{2}T_{ij}\right) + \left(\frac{1}{2}T_{ji} - \frac{1}{2}T_{ji}\right)$$

$$T_{ij} = \underbrace{\frac{1}{2}\left(T_{ij} + T_{ji}\right)}_{symmetric} + \underbrace{\frac{1}{2}\left(T_{ij} - T_{ji}\right)}_{antisymmetric}$$

Thus,

$$T_{(ij)} = \frac{1}{2}\left(T_{ij} + T_{ji}\right) \quad \text{and} \quad T_{[ij]} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

(d) It is also possible to show that the antisymmetric component of $\underline{\underline{T}}$ can be calculated as

$$T_{[ij]} = \frac{1}{2}\epsilon_{ijk}\epsilon_{klm}T_{lm}$$

(e) The scalar product of any symmetric and antisymmetric tensor is zero (Proof is assigned as Problem 3).

$$\underline{\underline{Q}} : \underline{\underline{R}} = \underline{\underline{R}} : \underline{\underline{Q}} = 0 \quad \text{if} \quad Q_{ij} = Q_{ji} \quad \text{and} \quad R_{ij} = -R_{ji}$$

(f) A more general form of the previous relationship can be stated as follows. The expression $A_{ijkl\ldots}B_{jklm\ldots} = 0$ is equal to zero if $\underline{\underline{\text{A}}}$ and $\underline{\underline{\text{B}}}$ are symmetric and antisymmetric (respectively) with respect to the same indices. For example,

$$A_{ijkl}B_{jklm} = 0$$

if $A_{ijkl} = A_{ikjl}$ and $B_{jklm} = -B_{kjlm}$. (The proof for this is assigned as Problem 4).

Problems

1. Write the following vector expressions in Cartesian Index Notation:
   (a) $\vec{a}\cdot\nabla\vec{a}$
   (b) $(\vec{a}\cdot\nabla)\vec{a}$
   (c) $\vec{a}\,\nabla\vec{a}$
   (d) $\underline{\underline{A}}:\underline{\underline{A}}$
   (e) $\underline{\underline{A}}:\underline{\underline{A}}^T$
   (f) curl grad div$(\vec{a}\times\vec{b})$

2. Is the expression "curl $[\vec{a}\cdot(\text{grad }\phi)]$" valid? Why or why not?

3. Show that $\underline{\underline{Q}}:\underline{\underline{R}}=\underline{\underline{R}}:\underline{\underline{Q}}=0$ if $\underline{\underline{Q}}$ is symmetric and $\underline{\underline{R}}$ is antisymmetric.

4. Show that $A_{ijkl}B_{jklm}=0$ if $\underline{\underline{A}}$ is symmetric with respect to indices j and k, and $\underline{\underline{B}}$ is antisymemtric with respect to j and k. Note that this is a more general version of problem 3.

5. Show that $\nabla\cdot\rho\vec{u}\vec{u}=\rho\vec{u}\cdot\nabla\vec{u}+\rho\vec{u}(\nabla\cdot\vec{u}\,)$

6. Show that $a_i\partial_j a_i=\nabla\left(\frac{1}{2}\vec{a}\cdot\vec{a}\right)$

7. Show that curl (grad ϕ)=0

8. Show that $\vec{a}\times(\vec{b}\times\vec{c}\,)=(\vec{a}\cdot\vec{c}\,)\vec{b}-(\vec{a}\cdot\vec{b}\,)\vec{c}$

9. Show that $(\vec{a}\times\vec{b}\,)\cdot(\vec{c}\times\vec{d}\,)=(\vec{a}\cdot\vec{c}\,)(\vec{b}\cdot\vec{d}\,)-(\vec{a}\cdot\vec{d}\,)(\vec{b}\cdot\vec{c}\,)$

10. Show that $\nabla\cdot(\nabla\times\vec{a}\,)=0$

11. Show that $\nabla\cdot(\vec{a}\times\vec{b}\,)=\vec{b}\cdot(\nabla\times\vec{a}\,)-\vec{a}\cdot(\nabla\times\vec{b}\,)$

12. Show that $\nabla\times(\vec{a}\times\vec{b}\,)=\vec{a}(\nabla\cdot\vec{b}\,)+\vec{b}\cdot\nabla\vec{a}-\vec{a}\cdot\nabla\vec{b}-\vec{b}(\nabla\cdot\vec{a}\,)$

13. Show that if div $\vec{u}=0$, div $\vec{v}=0$, and curl $\vec{w}=0$ then

$$\nabla\cdot[(\vec{u}\times\vec{v}\,)\times\vec{w}]=\vec{w}\cdot[(\vec{v}\cdot\nabla\,)\vec{u}-(\vec{u}\cdot\nabla\,)\vec{v}]$$

14. Show that $\nabla \times \left[\nabla\left(\frac{1}{2}\vec{a}\cdot\vec{a}\,\right) - \vec{a}\times(\nabla\times\vec{a}\,)\right] = \nabla\times(\vec{a}\cdot\nabla\vec{a}\,)$

15. Using the identity given in problem 14, show that

$$\nabla\times(\vec{a}\cdot\nabla\vec{a}\,) = \vec{a}\cdot\nabla(\nabla\times\vec{a}\,) + (\nabla\cdot\vec{a}\,)(\nabla\times\vec{a}\,) - (\nabla\times\vec{a}\,)\cdot(\nabla\vec{a})$$